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DETERMINATION OF THE INITIAL VELOCITY AND DURATION OF MOTION OF A FLUID IN CAPILLARIES

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The method of determination of the initial velocity and duration of motion of a fluid in capillaries which is based on reduction of the boundary-value problem under study to Cauchy problems for systems of ordinary differential equations is presented.

Solution of many applied problems of practical importance [1] is associated with solution of the nonlinear equation of motion of a fluid [2] in a capillary

$$U \frac{d^2U}{dt^2} + \left(\frac{dU}{dt}\right)^2 + \frac{8\eta}{r^2\rho} U \frac{dU}{dt} = \frac{2\sigma \cos \Theta}{r\rho} - gU \sin \varphi , \qquad (1)$$

where φ is the angle of inclination of the cylindrical capillary of radius r to the horizon.

In [3], a solution is presented for the case where the first two terms of Eq. (1), which account for the inertial force, are much smaller than the remaining terms in order of magnitude; therefore, they can be neglected.

A rather simple analytical solution of Eq. (1) is given in [4] for a horizontal capillary ($\varphi = 0$) and zero gravity ($g = 0$).

The general nonlinear equation (1) can be solved only by approximate analytical or numerical methods. Solution of Eq. (1) by the perturbation method in a small parameter in the first approximation is given in [5] for a vertical capillary ($\varphi = 90^0$). It should be noted that practical implementation of the results from [5] causes certain inconveniences because of the fact that the solution is represented in implicit form. In [6], a modified Fourier method has been used to obtain a simple approximate solution of Eq. (1) allowing one to describe more accurately the process of capillary soaking of the fluid in the initial stage of its rise when the influence of the inertial forces is substantial.

The authors of the works indicated above were engaged in investigating the kinetics of capillary soaking, starting, as a rule, from the solution of the Cauchy problem for Eq. (1). With this traditional approach, the deficient value of the initial velocity of capillary soaking was determined theoretically from the hypothesis for the least possible singularity of Eq. (1), for example, so that the second derivative $\left.\frac{d^2U}{dt^2}\right|_{t=0}$ is the final quantity [5, 6]. In the present work, we pose the nonclassical problem for Eq. (1) which makes it possible not only to investigate the kinetics of capillary soaking but also to determine certain important parameters of the process simulated.

Let us consider additional conditions for Eq. (1). At the initial instant of time, we have

$$U(t) = U_0 \quad \text{for} \quad t = 0 , \quad U_0 \neq 0 . \qquad (2)$$

If we denote the unknown time of motion of the fluid in the capillary until it stops completely by T, the following boundary condition is appropriate:

$$\frac{dU}{dt} = 0 \quad \text{for} \quad t = T . \qquad (3)$$

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On the other hand, if the maximum movement of the fluid in the capillary $U_{\max}$ is known, we have the boundary condition

$$U(t) = U_{\max} \quad \text{for} \quad t = T . \tag{4}$$

The value of $U_{\max} = 2\sigma \cos \Theta/(\rho r g \sin \varphi)$ was used, for example, in [6] in solving the Cauchy problem for Eq. (1) by the modified Fourier method.

Below we present a method of determination of the kinetics of capillary soaking on the basis of solution of the nonlinear boundary-value problem (1)–(4) in which the initial velocity $\left.\frac{dU}{dt}\right|_{t=0}$ and the time T can be determined simultaneously.

In solving the problem posed, we will use the idea of the quasilinearization method [7].

The quasilinear form of the nonlinear term $(dU/dt)^2$ in Eq. (1) is as follows:

$$\left(\frac{d\overset{(n+1)}{U}}{dt}\right)^2 \approx \left(\frac{d\overset{(n)}{U}}{dt}\right)\left[2\left(\frac{d\overset{(n+1)}{U}}{dt}\right) - \left(\frac{d\overset{(n)}{U}}{dt}\right)\right], \tag{5}$$

where $\overset{(n)}{U}$ is the nth approximation of the solution of (1)–(4).

Then the quasilinear equation corresponding to Eq. (1) will take the form

$$\frac{d^2\overset{(n+1)}{U}}{dt^2} + \left(a + \frac{2}{\overset{(n)}{U}}\frac{d\overset{(n)}{U}}{dt}\right)\frac{d\overset{(n+1)}{U}}{dt} + \frac{q}{\overset{(n)}{U}}\overset{(n+1)}{U} = \frac{1}{\overset{(n)}{U}}\left[f + \left(\frac{d\overset{(n)}{U}}{dt}\right)^2\right], \tag{6}$$

where $q = g \sin \varphi$, $a = 8\eta/(\rho r^2)$, and $f = 2\sigma \cos \Theta/(\rho r)$.

We note that Eq. (6) is the linear equation relative to the $(n+1)$ approximation of the solution of the problem.

The boundary conditions for Eq. (6) will be written as

$$\overset{(n+1)}{U} = U_0 \quad \text{for} \quad t = 0 , \tag{7}$$

$$\overset{(n+1)}{U} = U_{\max} \quad \text{for} \quad t = T , \tag{8}$$

$$\frac{d\overset{(n+1)}{U}}{dt} = 0 \quad \text{for} \quad t = T . \tag{9}$$

Since the time T is unknown, the solution of the quasilinear boundary-value problem (6)–(9) will be sought by the method of superposition in the form

$$\overset{(n+1)}{U}(t) = \varphi(t) + \mu\psi(t) , \tag{10}$$

where φ and ψ are functions unknown as yet and μ is a constant to be determined.

Next, we will assume that the unknown constant μ is equal to the sought initial velocity of motion of the fluid in the capillary, i.e.,

$$\mu = \frac{d\overset{(n+1)}{U}(0)}{dt} . \tag{11}$$

Then, with account for Eqs. (6), (7), (10), and (11), we obtain that the unknown auxiliary functions $\varphi(t)$ and $\psi(t)$ can be found by solution of the following Cauchy problems for systems of ordinary differential equations:

$$\frac{d\varphi}{dt} = P(t)\,, \quad \varphi(0) = U_0\,, \quad \frac{dP}{dt} = \frac{1}{\overset{(n)}{U}}\left[f + \left(\frac{d\overset{(n)}{U}}{dt}\right)^2\right] - \left[a + \frac{2}{\overset{(n)}{U}}\left(\frac{d\overset{(n)}{U}}{dt}\right)\right]P - \frac{q}{\overset{(n)}{U}}\varphi\,, \quad P(0) = 0\,; \tag{12}$$

$$\frac{d\psi}{dt} = F(t)\,, \quad \psi(0) = 0\,, \quad \frac{dF}{dt} = -\left[a + \frac{2}{\overset{(n)}{U}}\left(\frac{d\overset{(n)}{U}}{dt}\right)\right]F - \frac{q}{\overset{(n)}{U}}\psi\,, \quad F(0) = 1\,. \tag{13}$$

With allowance for the representation of the solution of the problem sought in the form (10), from boundary conditions (8) and (9) at $t = T$ we correspondingly will have

$$\varphi(T) + \mu\psi(T) = U_{\max}\,, \tag{14}$$

$$P(T) + \mu F(T) = 0\,. \tag{15}$$

Taking into consideration the notation adopted in Eq. (11), from Eq. (14) we obtain that the sought value of the initial velocity of motion of the fluid in the capillary is determined by the formula

$$\left.\frac{dU}{dt}\right|_{t=0} = \frac{U_{\max} - \varphi(T)}{\psi(T)}\,. \tag{16}$$

Then from Eq. (15) with account for Eq. (16) we obtain the equation

$$(\varphi(T) - U_{\max})\,F(T) - \psi(T)\,P(T) = 0\,, \tag{17}$$

which allows us to determine the time T.

After determination of the value of the constant $\mu = dU(0)/dt$ and the functions $\varphi(t)$ and $\psi(t)$ by solutions of the Cauchy problems (12) and (13), Eq. (1) is solved by the superposition formula (10).

Thus, the algorithm of solution of the initial problem consists of the following steps:

1. We select the initial approximation $\overset{(0)}{U}$, setting it equal, for example, to the solution of a "truncated" equation [3].

2. Using the well-known scheme of the Runge–Kutta method of fourth order of accuracy [8], we integrate the systems of ordinary differential equations of first order (12) and (13) with a step τ beginning with the time $t = 0$. During the integration, we check condition (17) at each step and finish the integration when this condition is satisfied. At the same time, we find the unknown duration T of motion of the fluid in the capillary until it stops completely and also the functions $\varphi(T)$ and $\psi(T)$.

3. From formula (16) we determine the value of the initial velocity $dU(0)/dt$.

4. According to Eq. (10), we set up the superposition of the solutions obtained and find $\overset{(1)}{U}_k$, $k = 1, 2, 3, \ldots, N$, where $\overset{(n)}{U}_k = \overset{(n)}{U}(k\tau)$.

5. The iteration process of solution of the problem posed is continued until the required exactness of the solution of the problem is achieved, i.e., until the condition $\max_k \left|\overset{(n+1)}{U}_k - \overset{(n)}{U}_k\right| \le \varepsilon$ is satisfied.

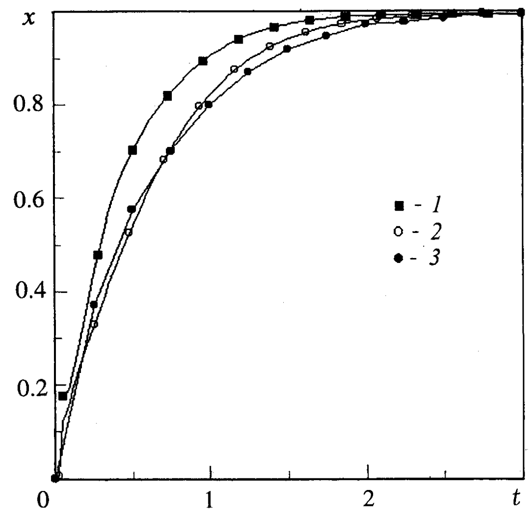

Fig. 1. Kinetics of capillary soaking of distilled water in a gravitational field: 1) calculation from Eq. (10); 2) calculation with the use of the calculated initial velocity from Eq. (16); 3) experimental data (r = 0.22·10^{-3} m). $x = U/U_{max}$ is dimensionless; t, sec.

To determine the efficiency of the method proposed for calculating the characteristics of fluid motion in capillaries, we carried out a computational experiment with the use of the experimental data of [6]; some of these results are presented in Fig. 1.

Using Eq. (10), the kinetics of capillary soaking was determined by solution of the nonlinear boundary-value problem (6)–(9) according to the procedure presented above (curve 1).

The duration of motion of the fluid T was determined from Eq. (17). The error in the calculated value of T did not exceed 10% for $\varepsilon = 10^{-5}$. Because of the absence of experimental data, we did not evaluate the error in the values of the initial velocity μ calculated from formula (16). However, using the value of the calculated parameter μ as the initial condition and by solution of the Cauchy problem (6), (7), and (11) according to the scheme of the Runge–Kutta method, we determined the kinetics of capillary soaking (curve 2). It is shown that the theoretical curves of capillary soaking are in qualitative agreement with the experimental curve. Here the curve (describing the kinetics of capillary soaking) calculated from Eq. (10) differs from the corresponding experimental curve (curve 3) to a greater extent than that calculated on the basis of model (1), (2), and (11) with account for the value of the initial velocity μ found using Eq. (16).

The analysis of the results of the performed computational experiment has shown that the present method can be used for establishing the mechanism of the process of capillary soaking and for determining its parameters.

NOTATION

U, length of movement of the meniscus; U_0, depth of immersion of the capillary in the fluid; U_{max}, maximum movement of the fluid in the capillary; Θ, wetting angle; g, free-fall acceleration; ρ, density; η, viscosity; σ, surface tension of the fluid; t, running time; r, radius; μ, initial velocity of motion of the fluid in the capillary; ε, required exactness of solution of the problem; τ, time step of numerical integration; N, number of steps of numerical integration required for determining the duration of motion of the fluid in the capillary with a prescribed accuracy; k, number of a node on the grid of numerical integration. Subscripts: max, maximum.

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